For the next few classes:

(work / $k = \bar{k}$)

The degree of a divisor on a curve

Let C be a smooth projective curve / $k = \bar{k}$.

Let

$$E = \sum m_i x_i \qquad (x_i \in C,\ m_i \in \mathbb{Z})$$

Define. $\deg E \equiv \sum m_i \in \mathbb{Z}$.

Proposition. Let C, D be smooth proj curves w/ a finite map:

$$\pi : C \longrightarrow D$$

(A) We have:

$$\deg(\pi^* E) = (\deg \pi)(\deg E)$$

(B) If $f \in k(C)^*$ is a meromorphic fn, then

$$\deg(\operatorname{div}(f)) = 0.$$

(So! deg is independent of rational equivalence. Given:

$$\deg : \operatorname{Pic} C \longrightarrow \mathbb{Z}.$$
)

Proof. (A) Suffices to prove for any $x \in D$ that

$$\deg(\pi^* x) = \deg(\pi).$$

We have:

- $\pi_* \mathcal{O}_C$ is finite + torsion-free on D.

$$\Downarrow$$

locally-free & $\operatorname{rk}(\pi_* \mathcal{O}_C) = [k(C) : k(D)]$

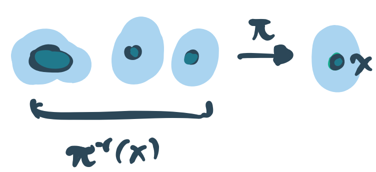
$$= \deg(\pi).$$

- Let $x \in D$ & (R, m) the local ring at x.

Let S be the semilocal ring of C at $\pi^{-1}(x)$. Then

$$R \subset S$$

is finite, $S \cong (\pi_* \mathcal{O}_C)_x$.



- $\pi^*(x) := \pi^{-1}(x)$

$$\cong \operatorname{Spec}(S / m \cdot S) \subset C.$$

So we want

$$\deg[\operatorname{Spec}(S/m.S)] \subset C = \operatorname{length}_k(S/mS)$$

$$= \dim_k\left((\pi_* \mathcal{O}_C) \otimes k(x)\right) = \big[k(C) : k(D)\big]$$

$$= \deg(\pi)$$

B A meromorphic fn $f \in k(C)^*$ gives a nonconstant map:

$$\begin{tikzcd} C \arrow[r, dotted] \arrow[dr, bend right, "f"'] & \mathbb{A}^1 \arrow[d, phantom, "\cap" rotate] \\ & \mathbb{P}^1_k . \end{tikzcd}$$

$$\operatorname{div}(f) = f^*[0] - f^*[\infty].$$

$$\Rightarrow \deg(\operatorname{div} f) = \deg f \cdot (0) = 0. \qquad \blacksquare$$

Cor If $\deg E < 0$ then $\Gamma(C, \mathcal{O}_C(E)) = 0$.

Riemann-Hurwitz.

Let $\pi : C \longrightarrow D$

(finite map of smooth curves)

Defn: Let $P \in C$. The **ramification index** of π at P is denoted e_P and defined as follows:

Let $x = \pi(P)$ w/ local ring $(\mathcal{O}_{D,x}, m_x = (t))$.

$$e_P := \dim_k \left(\mathcal{O}_{C,P} / \pi^* t \right) - 1.$$

The **ramification divisor** is defined as.

$$R = R_\pi = \sum e_P \cdot P. \quad \text{(if finite)}.$$

(Note: it is effective).

Assume char(k) = 0

Theorem Suppose $\pi^*\Omega_D \xrightarrow{d\pi} \Omega_C$ is nonzero $\Rightarrow$ injective. ($\equiv$ $k(C)/k(D)$ is separable)

Then: $\Omega_C \cong (\pi^*\Omega_D)\otimes \mathcal{O}_C(R)$.

($\Rightarrow \deg \Omega_C = (\deg \pi)\cdot(\deg \Omega_D) + \deg R$.)

Proof. $\pi^*\Omega_D$ & Ω_C are 2 line bundles on C. So given:

$$0 \to \pi^*\Omega_D \to \Omega_C \to Q \to 0.$$

$\Rightarrow$ Q is a skyscraper sheaf.

$\Rightarrow$ If $E = \sum_{P\in C} \operatorname{length}_P(Q)\cdot P = \sum_{P\in C} \ell_P \cdot P$

then: $\Omega_C \cong (\pi^*\Omega_D)\otimes \mathcal{O}(E)$.

Computation:

Let $P \in C$, $x = \pi(P)$.

$$(\mathcal{O}_{C,P}, m_P = (u)), \quad (\mathcal{O}_{D,x}, m_x = (t)).$$

Then: $\pi^*(t) = f\cdot u^{e_P+1}$.
$\qquad\qquad\uparrow$ unit

$$\Omega_{D,x} = (\mathcal{O}_{D,x})\cdot dt$$

$$\begin{array}{ccc}
(\pi^*\Omega_{D,x})_P & & \Omega_{C,P} \\
\wr\wr & & \wr\wr \\
\mathcal{O}_{C,P}\cdot dt & \xrightarrow[d\pi]{} & \mathcal{O}_{C,P}\cdot du
\end{array}$$

$$dt \longrightarrow d(f\cdot u^{e_P+1}) = f\,d(u^{e_P+1}) + u^{e_P+1}\,df.$$

$$= (e_p+1)u^{e_p} f\,du + u^{e_p+1} df.$$

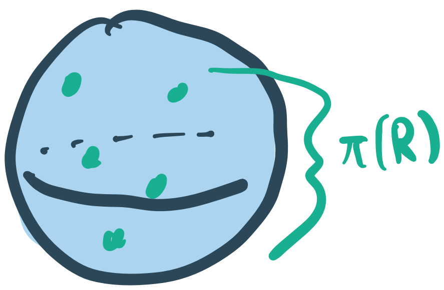
So: Image $= (u^{e_p})du. \Rightarrow \ell_p = e_p$. ■

EXAMPLE Let C be smooth & projective $/\mathbb{C}$.

Consider:

$$\pi : C \longrightarrow \mathbb{CP}^1.$$

This is a covering map away from $\pi(R)$.



So: $\chi_{top}(C) = \deg(\pi) \cdot \chi_{top}(\mathbb{CP}^1 \setminus \pi(R))$
$+ \#(\pi^{-1}(\pi(R)))$.

Check: For $x \in \mathbb{CP}^1$, $\{p_1, \dots, p_\ell\} = \pi^{-1}(x)$

then: $\sum_{i=1}^{\ell} e_{p_i} + 1 = \deg(\pi)$

So: $\ell + \sum e_{p_i} = \deg(\pi)$

$\Rightarrow \quad \ell = (\deg \pi) - \sum e_{p_i}$.

$\Rightarrow \chi_{top}(C) = \deg(\pi) \cdot \chi_{top}(\mathbb{CP}^1 \setminus \pi(R))$
$+ \left(\sum_{x \in \pi(R)} \deg \pi\right) - \deg R$

$$2-2g = \deg(\pi)\cdot(2) - \deg R.$$

$$\Rightarrow \deg(\pi)\cdot\deg(\Omega_{\mathbb{P}^1}) + \deg R = \deg \Omega_C$$

$$= 2g-2.$$

topological genus

The Hilbert Polynomial

Suppose $C \overset{i}{\subseteq} \mathbb{P}^n_k$ is a curve.

Let $L = i^*(\mathcal{O}(1))$.

Define the Hilbert function:

$$h_C(m) := \dim_k(\Gamma(C, L^{\otimes m})).$$

$$(= \dim_k(\Gamma(\mathbb{P}^n_k, i_* \mathcal{O}_C \otimes L^{\otimes m})))$$

Theorem. There is a linear function:

$$\ell(m) \in \mathbb{Z}[m]$$

such that $\ell(m) = h_C(m)$ for m sufficiently large.

Proof. By Bertini's theorem (overkill) there is a section:

$$s \in \Gamma(C, L)$$

such that $E = (s=0) \subset C$ is smooth
($\Rightarrow$ reduced!)

Consider the s.e.s.:

$$0 \longrightarrow \mathcal{O}_C \xrightarrow{s} L \longrightarrow L/s \longrightarrow 0$$

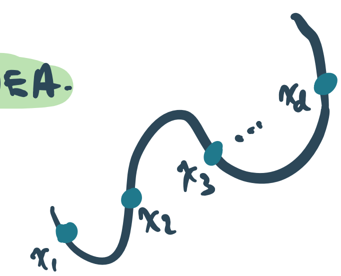
1. $L/s \cong L \otimes \mathcal{O}_E \cong \mathcal{O}_E$.
(skyscraper).

2. $\exists$ an m_0 s.t.

$$\Gamma(C, L^{\otimes m}) \longrightarrow \Gamma(C, \mathcal{O}_E)$$

is surjective $\forall\, m \geq m_0$.

IDEA.



$\Gamma(C, \mathcal{O}_E)$ has a basis of sections

$$e_i := \begin{cases} \neq 0 & \text{at } x_i \\ = 0 & \text{at } x_j \neq x_i. \end{cases}$$

We can prove "by hand" that it is surj. by considering linear polys:

$$l_j \in \Gamma(\mathbb{P}^n, \mathcal{O}(1)) \text{ s.t. } \begin{cases} l_j = 0 & \text{at } x_j \\ l_j \neq 0 & \text{at } x_n \neq x_j \end{cases}$$

Then $\ell_1 \cdot \ldots \cdot \hat{\ell}_i \cdot \ldots \cdot \ell_d \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d-1))$

vanishes at $x_j \neq x_i$.

Restricting to C works. ✓

3. For $m \geq d-1$ have:

$$0 \to \Gamma(C, L^{\otimes m}) \to \Gamma(C, L^{\otimes m+1}) \to \Gamma(C, \mathcal{O}_E) \to 0$$

exact!

$$\Rightarrow \dim \Gamma(C, L^{\otimes m+1}) = \dim(C, L^{\otimes m}) + d$$

So: $\ell(m) = d \cdot m + 1 - p$ works.

degree of C in $\mathbb{P}^n$ $=$ $\deg(L)$

"arithmetic genus" (some constant). ∎

Hilbert Serre Theorem: For any finitely generated graded module M ($/ k[x_0, \ldots, x_n]$), there is a polynomial $P_M(m)$ s.t.

$$P_M(m) = \dim_k(M_m) = \dim_k(\Gamma(\mathbb{P}^n_k, \tilde{M}(m)))$$

for m sufficiently large.

(Defn: Let C be a curve, $E \subset C$ a divisor.
$\ell(E) := \dim_k |E|$)

Riemann's Theorem

There is an integer g s.t. $\ell(E) \geq \deg(E) - g$ for all divisors E on C. The minimum such g is the "genus" of C.

Proof: W.T.S. $s(E) = \deg E - \ell(E)$ is bounded.

1. $s(0) = 0$.
2. If $E \equiv E'$, then $s(E) = s(E')$.
3. If $E \leq E'$ then $s(E) \leq s(E')$.

   Show for $E + p$.

   $$0 \to \mathcal{O}_C(E) \to \mathcal{O}_C(E+p) \to \mathcal{O}_p \to 0$$

   $$\Rightarrow \ell(E+p) \leq \ell(E) + 1.$$ ✓

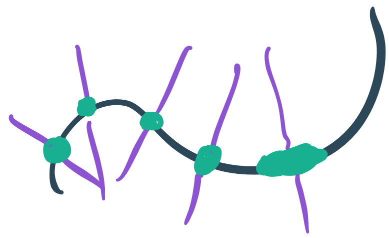
4. If E is very ample, then:

   $$s(mE) = m \cdot \deg E - (m \cdot \deg E + p)$$
   $$= p.$$

5. Show the bound p works.

For any E' eff w.T.S. $\exists\, m$ s.t.

$$E' \leq mE.$$

Proof:  $\blacksquare$

COR. The arithmetic genus is well defined.

COR. If $\ell(D) = \deg D - g$ then for all $D' \geq D$, $\ell(D') = \deg D' - g$.

(step 3. in the proof).

COR. $\exists\, N \in \mathbb{Z}$ s.t. for all D w/ $\deg D \geq N$

$$\ell(D) = \deg D - g.$$

Pf. $\exists$ some D' s.t. $\ell(D') = \deg D' - g$.

Let $N = \deg D' + g$.

If $\deg D \geq N$ then $\deg(D - D') - g \geq 1$. $\blacksquare$

Thus $D \geq D'$.

Riemann-Roch

Let K be a canonical divisor (i.e. $\mathcal{O}_C(K) = \omega_C$). Let D be a divisor on C:

$$\ell(D) = \deg D + 1 - g + \ell(K-D).$$

Remarks:

A. If $D = 0$ then we see:

$$\ell(K) = g - 1. \quad (\text{so } \dim_k(\Gamma(C, \omega_C)) = g).$$

B. If $\deg(K-D) < 0$

$\Rightarrow$ $K-D$ has no sections

$\Rightarrow$ $\ell(D) = \deg D + 1 - g \underbrace{-1}_{\text{convention when } \dim_k \Gamma(C, \mathcal{O}(D)) = 0}.$